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1993 J. Phys. A: Math. Gen. 26 5895

(http://iopscience.iop.org/0305-4470/26/21/028)

J. Phys. A: Math. Gen. 26 (1993) 5895–5903. Printed in the UK

A Backlund transformation and nonlinear superposition formula of a higher-order Ito equation

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Received 25 January 1993

Abstract. In this paper, we present a Backlund transformation for a higher-order Ito equation and prove a nonlinear superposition formula under certain conditions. Therefore, the integrability of the higher-order Ito equation is confirmed.

1. Introduction

The so-called Ito equation reads [1]

$$D_t(D_t+D_x^3)f\cdot f=0 \tag{1}$$

where the bilinear operator $D_x^m D_t^n$ is defined by [2]

$$D_x^m D_t^n a(x,t)\cdot b(x,t)\equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')\,|_{x'=x,t'=t}.$$

Until now, much research on this equation has been conducted [1, 3, 4]. In [1], Ito presented a higher-order version of (1)

$$(6D_tD_\tau+D_x^5D_\tau-5D_x^2D_\tau^2)f\cdot f=0 \tag{2a}$$

with

$$D_\tau(D_\tau+D_x^3)f\cdot f=0 \tag{2b}$$

and examined numerically that (2) has 3-soliton solution, where τ is an auxilary variable. In what follows, we refer to (2) as the higher-order Ito equation. To our knowledge, there are no further results on (2). Thus the integrability of the higher-order Ito equation remains to be confirmed.

In this paper, we present a Backlund transformation for (2) and prove a nonlinear superposition formula under certain conditions. Therefore, the integrability of the higher-order Ito equation (2) is confirmed.

2. A BT for the higher-order Ito equation

We obtain the following result:

Proposition. A BT for (2) is

$$(D_\tau+3\gamma^2 D_x-3\gamma D_x^2+D_x^3-\lambda)f\cdot f'=0 \tag{3a}$$

$$(D_xD_\tau-\mu D_x-\gamma D_\tau+\gamma\mu)f\cdot f'=0 \tag{3b}$$

$$(6D_t+D_x^5-5D_x^2D_\tau-5\gamma D_x^4-5\gamma^2D_x^3+5\lambda D_x^2+10\gamma\mu D_x-10\lambda\gamma D_x+k)f\cdot f'=0 \tag{3c}$$

where λ, μ, γ, k are arbitrary constants.

Proof. Let f be a solution of (2). If we can find three equations which relate f with f' such that

$$P_1\equiv D_\tau(D_\tau+D_x^3)f'\cdot f'=0$$
$$P_2\equiv(6D_tD_\tau+D_x^5D_\tau-5D_x^2D_\tau^2)f'\cdot f'=0$$

this is then a BT. Here we show that $(3a, b, c)$ indeed provides a BT for (2).

According to [4, 5], we know that $P_1=0$ can be proved in terms of $(3a, b)$. Thus it suffices to show that $P_2=0$. Making use of (A.1)–(A.6), $(3a, b, c)$, we have

$$\begin{aligned}
-f^2P_2={}&f'^2(6D_tD_\tau+D_x^5D_\tau-5D_x^2D_\tau^2)f\cdot f\\
&-f^2(6D_tD_\tau+D_x^5D_\tau-5D_x^2D_\tau^2)f'\cdot f'\\
&-5[D_\tau(D_\tau+D_x^3)f\cdot f](D_x^2f'\cdot f')+5[D_\tau(D_\tau+D_x^3)f'\cdot f'](D_x^2f\cdot f)\\
\overset{(A.1)-(A.3)}{=}{}&12D_\tau(D_tf\cdot f')\cdot ff'+2D_\tau(D_x^5f\cdot f')\cdot ff'\\
&+10D_\tau(D_x^2f\cdot f')\cdot(D_x^3f\cdot f')+10D_x^3(D_xf\cdot f')\cdot(D_xD_\tau f\cdot f')\\
&-10D_\tau[(D_x^2D_\tau f\cdot f')\cdot ff'+(D_\tau f\cdot f')\cdot(D_x^2f\cdot f')+2(D_xf\cdot f')\cdot(D_xD_\tau f\cdot f')]\\
\overset{(3a,b)}{=}{}&2D_\tau[(6D_t+D_x^5-5D_x^2D_\tau)f\cdot f']\cdot ff'\\
&+10D_\tau(D_x^2f\cdot f')\cdot[(-3\gamma^2D_x+3\gamma D_x^2+\lambda)f\cdot f']\\
&+10D_x^3(D_xf\cdot f')\cdot[(\mu D_x+\gamma D_\tau-\gamma\mu)f\cdot f']\\
&-20D_\tau(D_xf\cdot f')\cdot[(\mu D_x+\gamma D_\tau-\gamma\mu)f\cdot f']\\
\overset{(A.4)}{=}{}&2D_\tau[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda D_x^2+10\gamma\mu D_x)f\cdot f']\cdot ff'\\
&-30\gamma^2D_\tau(D_x^2f\cdot f')\cdot(D_xf\cdot f')+10\gamma D_x^3(D_xf\cdot f')\cdot(D_\tau f\cdot f')\\
&-10\gamma\mu D_x^3(D_xf\cdot f')\cdot ff'-20\gamma D_\tau(D_xf\cdot f')\cdot(D_\tau f\cdot f')\\
\overset{(A.5)}{=}{}&2D_\tau[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda D_x^2+10\gamma\mu D_x)f\cdot f']\cdot ff'\\
&-30\gamma^2D_\tau(D_x^2f\cdot f')\cdot(D_xf\cdot f')+10\gamma D_x^3(D_xD_\tau f\cdot f')\cdot ff'\\
&-10\gamma D_\tau[(D_x^4f\cdot f')\cdot ff'+2(D_xf\cdot f')\cdot(D_x^3f\cdot f')]\\
&-10\gamma\mu D_x^3(D_xf\cdot f')\cdot ff'-20\gamma D_\tau(D_xf\cdot f')\cdot(D_\tau f\cdot f')
\end{aligned}$$

$$\overset{(3a,b)}{=} 2D_\tau[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda D_x^2+10\gamma\mu D_x-5\gamma D_x^4)f\cdot f']\cdot ff'$$
$$-30\gamma^2 D_\tau(D_x^2f\cdot f')\cdot(D_xf\cdot f')+10\gamma D_x^3[(\gamma D_\tau-\gamma\mu)f\cdot f']\cdot ff'$$
$$-20\gamma D_\tau(D_xf\cdot f')\cdot[(-3\gamma^2D_x+3\gamma D_x^2+\lambda)f\cdot f']$$

$$\overset{(A.4)}{=} 2D_\tau[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda D_x^2+10\gamma\mu D_x-5\gamma D_x^4-10\lambda\gamma D_x)f\cdot f']\cdot ff'$$
$$-30\gamma^2D_\tau(D_x^2f\cdot f')\cdot(D_xf\cdot f')+10\gamma^2D_x^3(D_\tau f\cdot f')\cdot ff'$$
$$-60\gamma^2D_\tau(D_xf\cdot f')\cdot(D_x^2f\cdot f')$$
$$=2D_\tau[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda D_x^2+10\gamma\mu D_x-5\gamma D_x^4-10\lambda\gamma D_x)f\cdot f']\cdot ff'$$
$$+10\gamma^2D_x^3(D_\tau f\cdot f')\cdot ff'+30\gamma^2D_\tau(D_x^2f\cdot f')\cdot(D_xf\cdot f')$$

$$\overset{(A.6)}{=} 2D_\tau[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda D_x^2+10\gamma\mu D_x$$
$$-5\gamma D_x^4-10\lambda\gamma D_x+5\gamma^2D_x^3)f\cdot f']\cdot ff'$$
$$\overset{(3c)(A.4)}{=} 0$$

Thus we have completed the proof of the proposition.

3. Nonlinear superposition formula of the higher-order Ito equation

In this section, under certain conditions, we establish nonlinear superposition formula of the higher-order Ito equation. In the following discussion, we set $\gamma=k=0$ in BT (3) for the sake of convenience in calculation. In this case (3) becomes

$$(D_\tau+D_x^3-\lambda)f\cdot f'=0 \tag{3a'}$$
$$(D_xD_\tau-\mu D_x)f\cdot f'=0 \tag{3b'}$$
$$(6D_t+D_x^5-5D_x^2D_\tau+5\lambda D_x^2)f\cdot f'=0. \tag{3c'}$$

Let f_0 be a solution of the higher-order Ito equation (2), $f_0\neq 0$. Suppose that $f_i(i=1,2)$ is a solution of (2) which is related to f_0 under BT (3′) with (λ_i,μ_i), i.e. $f_0\xrightarrow{(\lambda_i,\mu_i)}f_i(i=1,2)$, and that f_{12} is defined by

$$D_xf_0\cdot f_{12}=cD_xf_1\cdot f_2 \qquad \text{(where } c \text{ is a non-zero constant)} \tag{4}$$

From these assumptions, we deduce that [4]

$$D_\tau f_1\cdot f_2+(\mu_1-\mu_2)f_1f_2+\frac{1}{c}D_\tau f_0\cdot f_{12}-\frac{1}{c}(\mu_1+\mu_2)f_0f_{12}=c_1(t,\tau)f_0^2 \tag{5}$$

$$D_\tau f_0\cdot f_{12}+\frac{1}{4}D_x^3f_0\cdot f_{12}+\frac{3c}{4}D_x^3f_1\cdot f_2-(\lambda_1+\lambda_2)f_0f_{12}=c_2(t,\tau)f_0^2 \tag{6}$$

$$\frac{1}{4}D_x^3f_1\cdot f_2+D_\tau f_1\cdot f_2-(\lambda_2-\lambda_1)f_1f_2+\frac{3}{4c}D_x^3f_0\cdot f_{12}=0 \tag{7}$$

where $c_i(t,\tau)(i=1,2)$ is some function of t, τ. Here and in the following, we assume that there exists a f_{12} determined by (4) such that $c_1(t,\tau)=0$ in (5) and $c_2(t,\tau)=0$ in (6), i.e.

$$D_\tau f_1\cdot f_2+(\mu_1-\mu_2)f_1f_2+\frac{1}{c}D_\tau f_0\cdot f_{12}-\frac{1}{c}(\mu_1+\mu_2)f_0f_{12}=0 \tag{5'}$$

$$D_\tau f_0\cdot f_{12}+\frac{1}{4}D_x^3f_0\cdot f_{12}+\frac{3c}{4}D_x^3f_1\cdot f_2-(\lambda_1+\lambda_2)f_0f_{12}=0 \tag{6'}$$

which implies, by using (7), (6′), that

$$\frac{1}{2}D_x^3f_1\cdot f_2-D_\tau f_1\cdot f_2+(\lambda_2-\lambda_1)f_1f_2=\frac{1}{2c}D_x^3f_0\cdot f_{12}-\frac{1}{c}D_\tau f_0\cdot f_{12}+\frac{1}{c}(\lambda_1+\lambda_2)f_0f_{12}. \tag{8}$$

In this case, we have [4]

$$\begin{aligned}
&(D_xD_\tau-\mu_2D_x)f_1\cdot f_{12}=0\\
&(D_xD_\tau-\mu_1D_x)f_2\cdot f_{12}=0\\
&(D_\tau+D_x^3-\lambda_2)f_1\cdot f_{12}=0\\
&(D_\tau+D_x^3-\lambda_1)f_2\cdot f_{12}=0.
\end{aligned}$$

Next, we have

$$\begin{aligned}
0=&[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda_1D_x^2)f_0\cdot f_1]f_2-[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda_2D_x^2f_2]f_1\\
\overset{(A.7-A.10)}{=}&-6f_0D_tf_1\cdot f_2-5f_{0_{xxxx}}D_xf_1\cdot f_2+10f_{0_{xxx}}(D_xf_1\cdot f_2)_x\\
&-\frac{5}{2}f_{0_{xx}}[D_x^3f_1\cdot f_2+3(D_xf_1\cdot f_2)_{xx}]+\frac{5}{2}f_{0_x}[D_x^3f_1\cdot f_2+(D_xf_1\cdot f_2)_{xx}]_x\\
&-\frac{1}{16}f_0[D_x^5f_1\cdot f_2+10(D_x^3f_1\cdot f_2)_{xx}+5(D_xf_1\cdot f_2)_{xxxx}]\\
&+10f_{0_{x\tau}}D_xf_1\cdot f_2+5f_{0_{xx}}D_\tau f_1\cdot f_2-5f_{0_x}[(D_xf_1\cdot f_2)_\tau+(D_\tau f_1\cdot f_2)_x]\\
&-5f_{0_\tau}(D_xf_1\cdot f_2)_x+\frac{5}{4}f_0[D_x^2D_\tau f_1\cdot f_2+(D_\tau f_1\cdot f_2)_{xx}+2(D_xf_1\cdot f_2)_{x\tau}]\\
&-5(\lambda_2-\lambda_1)f_{0_{xx}}f_1f_2-5f_{0_x}[(\lambda_1+\lambda_2)D_xf_1\cdot f_2+(\lambda_1-\lambda_2)(f_1f_2)_x]\\
&+\frac{5}{2}(\lambda_1+\lambda_2)f_0(D_xf_1\cdot f_2)_x+\frac{5}{4}(\lambda_1-\lambda_2)f_0[D_x^2f_1\cdot f_2+(f_1f_2)_{xx}]\\
\overset{(4,8)}{=}&f_0\Big\{-6D_tf_1\cdot f_2-\frac{1}{16}D_x^5f_1\cdot f_2\\
&-\frac{5}{4}\left[\frac{1}{2c}D_x^3f_0\cdot f_{12}-\frac{1}{c}D_\tau f_0\cdot f_{12}+\frac{1}{c}(\lambda_1+\lambda_2)f_0f_{12}\right]
\end{aligned}$$

$$-\frac{5}{16c}(D_xf_0\cdot f_{12})_{xxxx}+\frac{5}{2c}(\lambda_1+\lambda_2)(D_xf_0\cdot f_{12})_x+\frac{5}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2$$

$$+\frac{5}{4}D_x^2D_\tau f_1\cdot f_2+\frac{5}{2c}(D_xf_0\cdot f_{12})_{x\tau}\Big\}$$

$$-\frac{5}{c}f_{0_{xxxx}}D_xf_0\cdot f_{12}+\frac{10}{c}f_{0_{xxx}}(D_xf_0\cdot f_{12})_x$$

$$-5f_{0_{xx}}\left[\frac{1}{2c}D_x^3f_0\cdot f_{12}-\frac{1}{c}D_\tau f_0\cdot f_{12}+\frac{1}{c}(\lambda_1+\lambda_2)f_0f_{12}\right]$$

$$-\frac{15}{2c}f_{0_{xx}}(D_xf_0\cdot f_{12})_{xx}$$

$$+5f_{0_x}\left[\frac{1}{2c}D_x^3f_0\cdot f_{12}-\frac{1}{c}D_\tau f_0\cdot f_{12}+\frac{1}{c}(\lambda_1+\lambda_2)f_0f_{12}\right]_x$$

$$+\frac{5}{2c}f_{0_\tau}(D_xf_0\cdot f_{12})_{xxx}-\frac{5}{c}(\lambda_1+\lambda_2)f_{0_x}D_xf_0\cdot f_{12}+\frac{10}{c}f_{0_{x\tau}}D_xf_0\cdot f_{12}$$

$$-\frac{5}{c}f_{0_x}(D_xf_0\cdot f_{12})_\tau-\frac{5}{c}f_{0_\tau}(D_xf_0\cdot f_{12})_x$$

$$=f_0\Big\{-6D_tf_1\cdot f_2-\frac{1}{16}D_x^5f_1\cdot f_2+\frac{5}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2+\frac{5}{4}D_x^2D_\tau f_1\cdot f_2$$

$$-\frac{15}{4c}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}-\frac{15}{16c}D_x^5f_0\cdot f_{12}+\frac{15}{4c}D_x^2D_\tau f_0\cdot f_{12}\Big\}$$

which implies that

$$\begin{aligned}&-6D_tf_1\cdot f_2-\frac{1}{16}D_x^5f_1\cdot f_2+\frac{5}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2+\frac{5}{4}D_x^2D_\tau f_1\cdot f_2\\&-\frac{15}{4c}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}-\frac{15}{16c}D_x^5f_0\cdot f_{12}+\frac{15}{4c}D_x^2D_\tau f_0\cdot f_{12}=0.\end{aligned}\tag{9}$$

Moreover, from

$$[(D_\tau+D_x^3-\lambda_1)f_0\cdot f_1]_{xx}f_2-[(D_\tau+D_x^3-\lambda_2)f_0\cdot f_2]_{xx}f_1=0$$

we obtain, by using (A.11), (A.12), (4), (6′), (8)

$$-\frac{1}{16}D_x^5f_1\cdot f_2-\frac{1}{4}D_x^2D_\tau f_1\cdot f_2-\frac{1}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2+\frac{1}{16c}D_x^5f_0\cdot f_{12}$$

$$+\frac{1}{4c}D_x^2D_\tau f_0\cdot f_{12}-\frac{1}{4c}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}=0.\tag{10}$$

Further, from

$$[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda_1D_x^2)f_0\cdot f_1]_x f_2$$
$$-[(6D_t+D_x^5-5D_x^2D_\tau+5\lambda_2D_x^2)f_0\cdot f_2]_x f_1$$
$$+5[(D_\tau+D_x^3-\lambda_1)f_0\cdot f_1]_{xxx}f_2-5[(D_\tau+D_x^3-\lambda_2)f_0\cdot f_2]_{xxx}f_1=0$$

we obtain, by using (A.13)–(A.16), (4), (6′), (8)–(10)

$$f_{0x}\left[\frac{6}{c}D_tf_0\cdot f_{12}+\frac{1}{16c}D_x^5f_0\cdot f_{12}-\frac{5}{4c}D_x^2D_\tau f_0\cdot f_{12}-\frac{15}{4}D_x^2D_\tau f_1\cdot f_2\right.$$
$$\left.-\frac{15}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2+\frac{5}{4c}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}+\frac{15}{16}D_x^5f_1\cdot f_2\right]$$
$$+\frac{1}{2}f_0\left[-\frac{6}{c}D_tf_0\cdot f_{12}-\frac{1}{16c}D_x^5f_0\cdot f_{12}+\frac{5}{4c}D_x^2D_\tau f_0\cdot f_{12}+\frac{15}{4}D_x^2D_\tau f_1\cdot f_2\right.$$
$$\left.+\frac{15}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2-\frac{5}{4c}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}-\frac{15}{16}D_x^5f_1\cdot f_2\right]_x=0$$

which implies that

$$-\frac{6}{c}D_tf_0\cdot f_{12}-\frac{1}{16c}D_x^5f_0\cdot f_{12}+\frac{5}{4c}D_x^2D_\tau f_0\cdot f_{12}+\frac{15}{4}D_x^2D_\tau f_1\cdot f_2$$
$$+\frac{15}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2-\frac{5}{4c}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}-\frac{15}{16}D_x^5f_1\cdot f_2=c_3(t,\tau)f_0^2 \tag{11}$$

where $c_3(t, \tau)$ is some function of t, τ. Furthermore we assume that f_{12} determined by (4) is chosen such that $c_3(t, \tau)=0$, i.e.

$$-\frac{6}{c}D_tf_0\cdot f_{12}-\frac{1}{16c}D_x^5f_0\cdot f_{12}+\frac{5}{4c}D_x^2D_\tau f_0\cdot f_{12}+\frac{15}{4}D_x^2D_\tau f_1\cdot f_2$$
$$+\frac{15}{4}(\lambda_1-\lambda_2)D_x^2f_1\cdot f_2-\frac{5}{4c}(\lambda_1+\lambda_2)D_x^2f_0\cdot f_{12}-\frac{15}{16}D_x^5f_1\cdot f_2=0. \tag{12}$$

Then, similar to the deduction of (9), we can obtain

$$(6D_t+D_x^5-5D_x^2D_\tau+5\lambda_2D_2^2)f_1\cdot f_{12}=0$$
$$(6D_t+D_x^5-5D_x^2D_\tau+5\lambda_1D_x^2)f_2\cdot f_{12}=0.$$

Therefore f_{12} is a new solution of the higher-order Ito equation (2) which is related to f_1 and f_2.

To sum up, we can obtain some particular solutions via the following steps. First choose a given solution of the higher-order Ito equation (2). Second from the BT (3′), find f_1 and f_2 such that $f_0\xrightarrow{(\lambda_i,\mu_i)}f_i$ $(i=1, 2)$ and, further, obtain a particular solution $\tilde{f}_{12}$ from (4). Then a general solution of (4) is $f_{12}=k(t,\tau)f_0+\tilde{f}_{12}$ (where $k(t, \tau)$ is an arbitrary function of k, τ). Finally substitute f_{12} into (5), (6) and (11). If $k(t, \tau)$ can be determined such that $c_i(t, \tau)=0$ $(i=1, 2, 3)$, the corresponding f_{12} is a new solution of the higher-order Ito equation. For example, we have

$$\begin{array}{ccccc} & \xrightarrow{(0,p_1^3)} & 1+e^{\eta_1} & \searrow^{(0,p_2^3)} & \\ 1 & & & & \dfrac{p_1^3-p_2^3}{p_1^3+p_2^3}-e^{\eta_1}+e^{\eta_2}-\dfrac{p_1-p_2}{p_1+p_2}\,e^{\eta_1+\eta_2} \\ & \xrightarrow[(0,p_2^3)]{} & 1+e^{\eta_2} & \nearrow_{(0,p_1^3)} & \end{array}$$

where $\eta_i=p_i x-p_i^3\tau-p_i^5 t+\eta_i^0$, and p_i, η_i^0 are constants $(i=1, 2)$. So

$$\frac{p_1^3-p_2^3}{p_1^3+p_2^3}-e^{\eta_1}+e^{\eta_2}-\frac{p_1-p_2}{p_1+p_2}\,e^{\eta_1+\eta_2}$$

is a solution of (2).

$$\begin{array}{ccccc} & \xrightarrow{(0,p_3^3)} & \dfrac{p_1^3-p_3^3}{p_1^3+p_3^3}-e^{\eta_1}+e^{\eta_3}-\dfrac{p_1-p_3}{p_1+p_3}\,e^{\eta_1+\eta_3} & \searrow^{(0,p_2^3)} & \\ 1+e^{\eta_1} & & & & F \\ & \xrightarrow[(0,p_2^3)]{} & \dfrac{p_1^3-p_2^3}{p_1^3+p_2^3}-e^{\eta_1}+e^{\eta_2}-\dfrac{p_1-p_2}{p_1+p_2}\,e^{\eta_1+\eta_2} & \nearrow_{(0,p_3^3)} & \end{array}$$

where

$$\begin{aligned} F=&\frac{p_1^3-p_3^3}{p_1^3+p_3^3}-\frac{p_1^3-p_2^3}{p_1^3+p_2^3}-\frac{p_2^3-p_3^3}{p_2^3+p_3^3}-\frac{p_2^3-p_3^3}{p_2^3+p_3^3}\,e^{\eta_1} \\ &+\frac{p_1^3-p_3^3}{p_1^3+p_3^3}\,e^{\eta_2}-\frac{p_1^3-p_2^3}{p_1^3+p_2^3}\,e^{\eta_3}+\frac{p_1-p_2}{p_1+p_2}\,e^{\eta_1+\eta_2} \\ &+\frac{p_2-p_3}{p_2+p_3}\,e^{\eta_2+\eta_3}+\frac{p_3-p_1}{p_1+p_3}\,e^{\eta_1+\eta_3}+\frac{(p_1-p_3)(p_1-p_2)(p_2-p_3)}{(p_1+p_3)(p_1+p_2)(p_2+p_3)}\,e^{\eta_1+\eta_2+\eta_3} \end{aligned}$$

and $\eta_i=p_i x-p_i^3\tau-p_i^5 t+\eta_i^0$, p_i, η_i^0 are constants $(i=1, 2, 3)$. So F is a 3-soliton solution of (2), which confirms Ito's numerical result.

$$\begin{array}{ccccc} & \xrightarrow{(0,0)} & 1+x^2 & \searrow^{(0,p^3)} & \\ 1 & & & & -1-x^2+\left(-\dfrac{4}{p}\,x+1+x^2+\dfrac{4}{p^2}\right)e^{\eta} \\ & \xrightarrow[(0,p^3)]{} & 1+e^{\eta} & \nearrow_{(0,0)} & \end{array}$$

where $\eta=px-p^3\tau-p^5 t+\eta^0$, p, η^0 are constants. So

$$-1-x^2+\left(-\frac{4}{p}\,x+1+x^2+\frac{4}{p^2}\right)e^{\eta}$$

is also a solution of (2).

Acknowledgments

The author would like to express his sincere thanks to professor Tu Gui-Zhang for his guidance and encouragement. This work was supported by the National Natural Science Fund of China.

Appendix

The following bilinear operator identities hold for any arbitrary functions a, b, c and d:

$$(D_tD_ya\cdot a)b^2 - a^2D_tD_yb\cdot b = 2D_t(D_ya\cdot b)\cdot ab \tag{A.1}$$

$$\begin{aligned}(D_x^5D_ta\cdot a)b^2 - a^2D_x^5D_tb\cdot b + 5(D_x^3D_tb\cdot b)(D_x^2a\cdot a) - 5(D_x^3D_ta\cdot a)(D_x^2b\cdot b)\\ = 2D_t(D_x^5a\cdot b)\cdot ab + 10D_t(D_x^2a\cdot b)\cdot(D_x^3a\cdot b) + 10D_x^3(D_xa\cdot b)\cdot(D_xD_ta\cdot b)\end{aligned} \tag{A.2}$$

$$\begin{aligned}(D_x^2D_t^2a\cdot a)b^2 - a^2D_x^2D_t^2b\cdot b + (D_t^2a\cdot a)(D_x^2b\cdot b) - (D_t^2b\cdot b)(D_x^2a\cdot a)\\ = 2D_t[(D_x^2D_ta\cdot b)\cdot ab + (D_ta\cdot b)\cdot(D_x^2a\cdot b) + 2(D_xa\cdot b)\cdot(D_xD_ta\cdot b)]\end{aligned} \tag{A.3}$$

$$D_x^{2n+1}a\cdot a = 0 \tag{A.4}$$

$$D_x^3[(D_xD_ta\cdot b)\cdot ab + (D_ta\cdot b)\cdot(D_xa\cdot b)] = D_t[(D_x^4a\cdot b)\cdot ab + 2(D_xa\cdot b)\cdot(D_x^3a\cdot b)] \tag{A.5}$$

$$D_x^3(D_ta\cdot b)\cdot ab = D_t[(D_x^3a\cdot b)\cdot ab + 3(D_xa\cdot b)\cdot(D_x^2a\cdot b)] \tag{A.6}$$

$$(D_ta\cdot b)c - (D_ta\cdot c)b = -aD_tb\cdot c \tag{A.7}$$

$$\begin{aligned}(D_x^5a\cdot b)c - (D_x^5a\cdot c)b = {}& -5a_{xxxx}D_xb\cdot c + 10a_{xxx}(D_xb\cdot c)_x\\ & -\frac{5}{2}a_{xx}[D_x^3b\cdot c + 3(D_xb\cdot c)_{xx}] + \frac{5}{2}a_x[D_x^3b\cdot c + (D_xb\cdot c)_{xx}]_x\\ & -\frac{a}{16}[D_x^5b\cdot c + 10(D_x^3b\cdot c)_{xx} + 5(D_xb\cdot c)_{xxxx}]\end{aligned} \tag{A.8}$$

$$\begin{aligned}(D_x^2D_ya\cdot b)c - (D_x^2D_ya\cdot c)b = {}& -2a_{xy}D_xb\cdot c - a_{xx}D_yb\cdot c\\ & + a_x[(D_xb\cdot c)_y + (D_yb\cdot c)_x] + a_y(D_xb\cdot c)_x\\ & -\frac{a}{4}[D_x^2D_yb\cdot c + (D_yb\cdot c)_{xx} + 2(D_xb\cdot c)_{xy}]\end{aligned} \tag{A.9}$$

$$\begin{aligned}\lambda_1(D_x^2a\cdot b)c - \lambda_2(D_x^2a\cdot c)b = {}& (\lambda_1-\lambda_2)a_{xx}bc\\ & - a_x[(\lambda_1+\lambda_2)D_xb\cdot c + (\lambda_1-\lambda_2)(bc)_x] + a[\lambda_1b_{xx}c - \lambda_2bc_{xx}]\end{aligned} \tag{A.10}$$

$$\begin{aligned}(D_x^3a\cdot b)_{xx}c - (D_x^3a\cdot c)_{xx}b = {}& -a_{xxxx}D_xb\cdot c - 2a_{xxx}(D_xb\cdot c)_x\\ & +\frac{1}{2}a_{xx}[D_x^3b\cdot c + 3(D_xb\cdot c)_{xx}] + \frac{1}{2}a_x[D_x^3b\cdot c + (D_xb\cdot c)_{xx}]_x\\ & -\frac{a}{16}[D_x^5b\cdot c + 10(D_x^3b\cdot c)_{xx} + 5(D_xb\cdot c)_{xxxx}]\end{aligned} \tag{A.11}$$

$$(D_y a\cdot b)_{xx}c-(D_y a\cdot c)_{xx}b=2a_{xy}D_x b\cdot c-a_{xx}D_y b\cdot c$$
$$-a_x[(D_x b\cdot c)_y+(D_y b\cdot c)_x]+a_y(D_x b\cdot c)_x$$
$$-\frac{a}{4}[D_x^2 D_y b\cdot c+(D_y b\cdot c)_{xx}+2(D_x b\cdot c)_{xy}] \quad (A.12)$$

$$(D_x^5 a\cdot b)_x c-(D_x^5 a\cdot c)_x b+5(D_x^3 a\cdot b)_{xxx}c-5(D_x^3 a\cdot c)_{xxx}b$$
$$=-4a_{xxxxxx}D_x b\cdot c-10a_{xxxxx}(D_x b\cdot c)_x+5a_{xxx}[(D_x^3 b\cdot c)_x+(D_x b\cdot c)_{xxx}]$$
$$+\frac{1}{4}a_x[D_x^5 b\cdot c+10(D_x^3 b\cdot c)_{xx}+5(D_x b\cdot c)_{xxxx}]$$
$$-\frac{3}{8}a[3(D_x^5 b\cdot c)_x+10(D_x^3 b\cdot c)_{xxx}+3(D_x b\cdot c)_{xxxxx}] \quad (A.13)$$

$$(D_t a\cdot b)_x c-(D_t a\cdot c)_x b=a_t D_x b\cdot c-a_x D_t b\cdot c-\frac{1}{2}a[(D_x b\cdot c)_t+(D_t b\cdot c)_x] \quad (A.14)$$

$$\lambda_1(D_x^2 a\cdot b)_x c-\lambda_2(D_x^2 a\cdot c)_x b-\lambda_1(ab)_{xxx}c+\lambda_2(ac)_{xxx}b$$
$$=-2a_x\left\{(\lambda_1+\lambda_2)(D_x b\cdot c)_x+\frac{1}{2}(\lambda_1-\lambda_2)[D_x^2 b\cdot c+(bc)_{xx}]\right\}$$
$$-2a_{xx}[(\lambda_1+\lambda_2)D_x b\cdot c+(\lambda_1-\lambda_2)(bc)_x] \quad (A.15)$$

$$(D_x^2 D_y a\cdot b)_x c-(D_x^2 D_y a\cdot c)_x b-(D_y a\cdot b)_{xxx}c+(D_y a\cdot c)_{xxx}b$$
$$=-4a_{xxy}D_x b\cdot c-4a_{xy}(D_x b\cdot c)_x+2a_{xx}[(D_x b\cdot c)_y+(D_y b\cdot c)_x]$$
$$+a_x[D_x^2 D_y b\cdot c+(D_y b\cdot c)_{xx}+2(D_x b\cdot c)_{xy}] \quad (A.16)$$

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